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Oscillation criteria for a generalized Emden-Fowler dynamic equation on time scales

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Abstract

In this paper, we consider the second-order Emden-Fowler neutral delay dynamic equation

$$\left(r(t)\left|z^{\Delta}(t)\right|^{\alpha-1}z^{\Delta}(t)\right)^{\Delta}+q(t)\left|x\left(\delta(t)\right)\right|^{\beta-1}x\left(\delta(t)\right)=0$$

on time scales, where $z(t)=x(t)+p(t)x(\tau(t))$ and $\beta\geq\alpha>0$ are constants. By means of the Riccati transformation and inequality technique, some oscillation criteria are established, which extend and improve some known results in the literature.

MSC: 34K11; 34K40; 34N05

Keywords: oscillation; second-order; Emden-Fowler equation; neutral delay dynamic equation; time scale

1 Introduction

In this paper, we are concerned with the oscillation for the following generalized Emden-Fowler dynamic equations:

$$\left(r(t)\left|z^{\Delta}(t)\right|^{\alpha-1}z^{\Delta}(t)\right)^{\Delta}+q(t)\left|x\left(\delta(t)\right)\right|^{\beta-1}x\left(\delta(t)\right)=0, \tag{1.1}$$

where $t\in[t_0,\infty)_{\mathbb{T}}:=[t_0,\infty)\cap\mathbb{T}$ with $\sup\mathbb{T}=\infty$, $z(t):=x(t)+p(t)x(\tau(t))$, α and β are two constants.

Throughout this paper, we assume that:

(A1) $\beta\geq\alpha>0$;

(A2) $r\in C_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},(0,\infty))$, $\int_{t_0}^{\infty}(\frac{1}{r(t)})^{\frac{1}{\alpha}}\Delta t=\infty$;

(A3) $p\in C_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},[0,\infty))$, $q\in C_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},(0,\infty))$;

(A4) $\tau,\delta\in C^1_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{T})$, $\lim_{t\to\infty}\tau(t)=\lim_{t\to\infty}\delta(t)=\infty$.

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. On a time scale $\mathbb{T}$ we define the forward and backward jump operators by $\sigma(t):=\inf\{s\in\mathbb{T}|s>t\}$ and $\rho(t):=\sup\{s\in\mathbb{T}|s<t\}$, $\inf\emptyset:=\sup\mathbb{T}$, $\emptyset$ denotes the empty set. A point $t\in\mathbb{T}$ is said to be left-dense if $\rho(t)=t$ and $t>\inf\mathbb{T}$, right-dense if $\sigma(t)=t$ and $t<\sup\mathbb{T}$, left-scattered if $\rho(t)<t$, and right-scattered if $\sigma(t)>t$. Points that are right-scattered and left-scattered at the same time are called isolated. The graininess function $\mu : \mathbb{T} \to [0,\infty)$ is defined by $\mu(t) := \sigma(t) - t$ and for any function $f : \mathbb{T} \to \mathbb{R}$ the notation $f^\sigma(t) := f(\sigma(t))$. For some other concepts related to the notion of time scale, see [1, 2].

As a solution of (1.1), we mean a nontrivial real function x such that $x(t) + p(t)x(\tau(t)) \in C^1_{\mathrm{rd}}[t_x,\infty)$ and $r(t)|z^\Delta(t)|^{\alpha-1}z^\Delta(t) \in C^1_{\mathrm{rd}}[t_x,\infty)$ for a certain $t_x \geq t_0$ and satisfying (1.1) for $t \geq t_x$. Our attention is restricted to those solutions of (1.1) which exist on the half-line $[t_x,\infty)$ and satisfy $\sup\{|x(t)| : t > t_*\} > 0$ for any $t_* \geq t_x$. We recall that a solution x of equation (1.1) is said to be nonoscillatory if there exists a $t_0 \in \mathbb{T}$ such that $x(t)x(\sigma(t)) > 0$ for all $t \in [t_0,\infty)_\mathbb{T}$; otherwise, it is said to be oscillatory. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

In recent years, there has been an increasing interest in studying oscillatory behavior of second-order neutral delay dynamic equations on time scales, and many results have been obtained; see, for example, [3–7].

We known that the Emden-Fowler equation and its generalized forms have attracted extensive attention because of the relevance to nuclear physics and gaseous dynamics in astrophysics. Besides, the second order neutral delay differential equations are also used in many fields. Recently, many results have been obtained on the oscillation of these equations, we refer the reader to [8–16] and the references cited therein.

Liu *et al.* [15] studied the generalized Emden-Fowler equation

$$\bigl(r(t)\bigl|z'(t)\bigr|^{\alpha-1}z'(t)\bigr)' + q(t)\bigl|x\bigl(\sigma(t)\bigr)\bigr|^{\beta-1}x\bigl(\sigma(t)\bigr) = 0, \tag{1.2}$$

where $z(t) = x(t) + p(t)x(\tau(t))$, $0 \leq p(t) \leq 1$; α and β are two constants. By use of an averaging technique and specific analytical skills, some oscillation and asymptotic criteria are established.

Chen [16] considered the Emden-Fowler neutral delay dynamic equation

$$\bigl(r(t)\bigl|z^\Delta(t)\bigr|^{\alpha-1}z^\Delta(t)\bigr)^\Delta + f\bigl(t, y\bigl(\delta(t)\bigr)\bigr) = 0, \tag{1.3}$$

where $z(t) = y(t) + p(t)y(\tau(t))$, $\alpha > 0$ is a constant, and there exists a positive right-dense-continuous function $q(t)$ such that $|f(t,u)| \geq q(t)|u^\alpha|$. By applying a generalized Riccati transformation technique, they obtained several oscillation theorems for equation (1.3).

Our research in this paper is the extension of equation (1.2) on time scale, and we will derive several oscillation criteria of equation (1.1), respectively, for two cases, *i.e.*, $0 \leq p \leq 1$ and $p > 1$. Compared with equation (1.3), equation (1.1) contains another parameter β and it does not satisfy the assumption on function f in [16]; then the existing results cannot be applied to our equation.

All functional inequalities considered in this paper are assumed to hold eventually, that is, they are satisfied for all t large enough.

2 Some preliminaries and lemmas

We will make use of the following product and quotient rules for the derivative of the differentiable functions on time scales:

$$(fg)^\Delta(t) = f^\Delta(t)g(t) + f\bigl(\sigma(t)\bigr)g^\Delta(t) = f(t)g^\Delta(t) + f^\Delta(t)g\bigl(\sigma(t)\bigr), \tag{2.1}$$

$$\left(\frac{f}{g}\right)^\Delta(t) = \frac{f^\Delta(t)g(t) - f(t)g^\Delta(t)}{g(t)g(\sigma(t))}, \quad \text{where } gg^\sigma \neq 0. \tag{2.2}$$

For $b, c \in \mathbb{T}$ and a differentiable function f, the Cauchy integral of f^{Δ} is defined by

$$\int_b^c f^{\Delta}(t)\Delta t = f(c) - f(b).$$

The integration by parts formula reads

$$\int_b^c f^{\Delta}(t)g(t)\Delta t = f(c)g(c) - f(b)g(b) - \int_b^c f\bigl(\sigma(t)\bigr)g^{\Delta}(t)\Delta t,$$

and infinite integrals are defined by

$$\int_b^{\infty} f(s)\Delta s = \lim_{t\to\infty}\int_b^t f(s)\Delta s.$$

For more details, see [1].

Lemma 2.1 *Assume that $x(t)$ is an eventually positive solution of* (1.1). *Then there exists $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that*

$$\bigl(r(t)\bigl|z^{\Delta}(t)\bigr|^{\alpha-1}z^{\Delta}(t)\bigr)^{\Delta} < 0, \quad z^{\Delta}(t) > 0, t \in [t_1, \infty)_{\mathbb{T}}.$$

Proof Since $x(t)$ is an eventually positive solution of (1.1), there exists $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $x(t) > 0$, $x(\tau(t)) > 0$, $x(\delta(t)) > 0$ for all $t \in [t_1, \infty)_{\mathbb{T}}$. From the definition of $z(t)$ and (A3), we get

$$z(t) \geq x(t) > 0, \quad t \in [t_1, \infty)_{\mathbb{T}}. \tag{2.3}$$

It follows from (1.1), (2.3), and (A3) that

$$\bigl(r(t)\bigl|z^{\Delta}(t)\bigr|^{\alpha-1}z^{\Delta}(t)\bigr)^{\Delta} = -q(t)\bigl[x\bigl(\delta(t)\bigr)\bigr]^{\beta} < 0, \quad t \in [t_1, \infty)_{\mathbb{T}}.$$

Therefore $r(t)|z^{\Delta}(t)|^{\alpha-1}z^{\Delta}(t)$ is a strictly decreasing function on $[t_1, \infty)_{\mathbb{T}}$ and $z^{\Delta}(t)$ is eventually of one sign.

We claim that

$$z^{\Delta}(t) > 0, \quad t \in [t_1, \infty)_{\mathbb{T}}. \tag{2.4}$$

If not, then there exists $t_2 \in [t_1, \infty)_{\mathbb{T}}$ such that $z^{\Delta}(t) \leq 0$, $t \in [t_2, \infty)_{\mathbb{T}}$. Hence from (A2) we have $r(t)|z^{\Delta}(t)|^{\alpha-1}z^{\Delta}(t) \leq 0$, $t \in [t_2, \infty)_{\mathbb{T}}$. Since $r(t)|z^{\Delta}(t)|^{\alpha-1}z^{\Delta}(t)$ is strictly decreasing on $[t_1, \infty)_{\mathbb{T}}$, it is clear that $r(t_2)|z^{\Delta}(t_2)|^{\alpha-1}z^{\Delta}(t_2) < r(t_1)|z^{\Delta}(t_1)|^{\alpha-1}z^{\Delta}(t_1)$. Therefore, for $t \in [t_2, \infty)$, there exists a constant $c \geq 0$ such that

$$r(t)\bigl|z^{\Delta}(t)\bigr|^{\alpha-1}z^{\Delta}(t) \leq r(t_2)\bigl|z^{\Delta}(t_2)\bigr|^{\alpha-1}z^{\Delta}(t_2) = -c,$$

that is $-r(t)(-z^{\Delta}(t))^{\alpha} \leq -c$. Thus, we obtain

$$z^{\Delta}(t) \leq -c^{\frac{1}{\alpha}}\left(\frac{1}{r(t)}\right)^{\frac{1}{\alpha}}, \quad t \in [t_2, \infty)_{\mathbb{T}}.$$

Integrating both sides of the last inequality from t_2 to t, we get

$$z(t) - z(t_2) \leq -c^{\frac{1}{\alpha}} \int_{t_2}^{t} \left(\frac{1}{r(s)}\right)^{\frac{1}{\alpha}}, \quad t \in [t_2, \infty)_{\mathbb{T}}.$$

Noting (A2) and letting $t \to \infty$, we see that $\lim_{t\to\infty} z(t) = -\infty$. This contradicts the fact that $z(t) > 0$. Hence (2.4) holds. This completes the proof. □

Lemma 2.2 ([1], Theorem 1.90) *If x is differentiable, then*

$$\left(x^{\gamma}\right)^{\Delta} = \gamma x^{\Delta} \int_0^1 \left[h x^{\sigma} + (1-h)x\right]^{\gamma-1} dh,$$

where γ is a constant.

3 Oscillation of equation (1.1) when $0 \leq p(t) \leq 1$

Theorem 3.1 *Assume that* (A1)-(A4) *hold, and $\tau(t) \leq t$, $\delta(t) \leq t$ for $t \in [t_0, \infty)_{\mathbb{T}}$. If there exists a positive function $\varphi \in C^1_{\mathrm{rd}}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$ such that for any positive number M and sufficiently large $t_1 \geq t_0$ we have*

$$\limsup_{t\to\infty} \int_{t_2}^{t} \left[\varphi(s)Q(s)\psi^{\beta}(s, t_1) - \frac{\alpha^{\alpha} r(s)((\varphi^{\Delta}(s))_{+})^{\alpha+1}}{(\alpha+1)^{\alpha+1}(\beta M)^{\alpha}\varphi^{\alpha}(s)}\right] \Delta s = \infty, \tag{3.1}$$

where $t_2 > t_1$ such that $\delta(t) > t_1$ for $t \in [t_2, \infty)_{\mathbb{T}}$, $\psi(s, t_1) = (\int_{t_1}^{s} \frac{1}{r^{\frac{1}{\alpha}}(u)} \Delta u)^{-1} \int_{t_1}^{\delta(s)} \frac{1}{r^{\frac{1}{\alpha}}(u)} \Delta u$, $s \in [t_2, \infty)_{\mathbb{T}}$ and $Q(s) = q(s)[1 - p(\delta(s))]^{\beta}$, $(\varphi^{\Delta}(s))_{+} = \max\{\varphi^{\Delta}(s), 0\}$. Then equation (1.1) *is oscillatory.*

Proof Suppose that $x(t)$ is a nonoscillatory solution of (1.1). Without loss of generality, we may assume that $x(t) > 0$, $x(\tau(t)) > 0$, $x(\delta(t)) > 0$, $t \in [t_1, \infty)_{\mathbb{T}}$. From Lemma 2.1, we get $z^{\Delta}(t) > 0$, then $z(t)$ is a strictly increasing function on $[t_1, \infty)_{\mathbb{T}}$. Using the definition of z, we have

$$z(t) \leq x(t) + p(t)z\big(\tau(t)\big) \leq x(t) + p(t)z(t),$$

that is,

$$x(t) \geq \big(1 - p(t)\big)z(t).$$

Then we have

$$\big(x\big(\delta(t)\big)\big)^{\beta} \geq \big(1 - p\big(\delta(t)\big)\big)^{\beta}\big(z\big(\delta(t)\big)\big)^{\beta}.$$

Notice by the definition of $Q(t)$ and (1.1), we get

$$\big(r(t)\big(z^{\Delta}(t)\big)^{\alpha}\big)^{\Delta} + Q(t)\big(z\big(\delta(t)\big)\big)^{\beta} \leq 0. \tag{3.2}$$

Define a function

$$\omega(t) = \varphi(t)\frac{r(t)(z^{\Delta}(t))^{\alpha}}{z^{\beta}(t)}, \quad t \in [t_1, \infty)_{\mathbb{T}}. \tag{3.3}$$

Obviously, $\omega(t) > 0$. From (2.1), (2.2), (3.2), and (3.3), we have

$$\begin{aligned}\omega^{\Delta} &= \bigl(r\bigl(z^{\Delta}\bigr)^{\alpha}\bigr)^{\Delta}\frac{\varphi}{z^{\beta}} + \bigl(r\bigl(z^{\Delta}\bigr)^{\alpha}\bigr)^{\sigma}\left(\frac{\varphi^{\Delta}}{(z^{\beta})^{\sigma}} - \frac{\varphi(z^{\beta})^{\Delta}}{z^{\beta}(z^{\beta})^{\sigma}}\right) \\ &\leq -Q(z\circ\delta)^{\beta}\frac{\varphi}{z^{\beta}} + \frac{(\varphi^{\Delta})_{+}}{\varphi^{\sigma}}\omega^{\sigma} - \frac{\varphi}{\varphi^{\sigma}}\frac{(z^{\beta})^{\Delta}}{z^{\beta}}\omega^{\sigma}. \end{aligned} \tag{3.4}$$

Since $r(t)(z^{\Delta}(t))^{\alpha}$ is strictly decreasing on $[t_1,\infty)_{\mathbb{T}}$, we get

$$\begin{aligned} z(t) - z\bigl(\delta(t)\bigr) &= \int_{\delta(t)}^{t}\frac{(r(u)(z^{\Delta}(u))^{\alpha})^{\frac{1}{\alpha}}}{r^{\frac{1}{\alpha}}(u)}\Delta u \\ &\leq \bigl(r\bigl(\delta(t)\bigr)\bigl(z^{\Delta}\bigl(\delta(t)\bigr)\bigr)^{\alpha}\bigr)^{\frac{1}{\alpha}}\int_{\delta(t)}^{t}\frac{1}{r^{\frac{1}{\alpha}}(u)}\Delta u, \quad t\in[t_1,\infty)_{\mathbb{T}}. \end{aligned} \tag{3.5}$$

Taking $t_2\in[t_1,\infty)_{\mathbb{T}}$ such that $\delta(t) > t_1$ for $t\in[t_2,\infty)_{\mathbb{T}}$, we obtain

$$\begin{aligned} z\bigl(\delta(t)\bigr) > z\bigl(\delta(t)\bigr) - z(t_1) &= \int_{t_1}^{\delta(t)}\frac{(r(u)(z^{\Delta}(u))^{\alpha})^{\frac{1}{\alpha}}}{r^{\frac{1}{\alpha}}(u)}\Delta u \\ &\geq \bigl(r\bigl(\delta(t)\bigr)\bigl(z^{\Delta}\bigl(\delta(t)\bigr)\bigr)^{\alpha}\bigr)^{\frac{1}{\alpha}}\int_{t_1}^{\delta(t)}\frac{1}{r^{\frac{1}{\alpha}}(u)}\Delta u. \end{aligned} \tag{3.6}$$

For $t\in[t_2,\infty)_{\mathbb{T}}$, from (3.5), (3.6) we obtain

$$\frac{z(t)}{z(\delta(t))} \leq 1 + \frac{(r(\delta(t))(z^{\Delta}(\delta(t)))^{\alpha})^{\frac{1}{\alpha}}}{z(\delta(t))}\int_{\delta(t)}^{t}\frac{1}{r^{\frac{1}{\alpha}}(u)}\Delta u$$

and

$$\frac{(r(\delta(t))(z^{\Delta}(\delta(t)))^{\alpha})^{\frac{1}{\alpha}}}{z(\delta(t))} < \left(\int_{t_1}^{\delta(t)}\frac{1}{r^{\frac{1}{\alpha}}(u)}\Delta u\right)^{-1}.$$

Therefore

$$\begin{aligned} \frac{z(t)}{z(\delta(t))} &\leq 1 + \left(\int_{t_1}^{\delta(t)}\frac{1}{r^{\frac{1}{\alpha}}(u)}\Delta u\right)^{-1}\int_{\delta(t)}^{t}\frac{1}{r^{\frac{1}{\alpha}}(u)}\Delta u \\ &= \int_{t_1}^{t}\frac{1}{r^{\frac{1}{\alpha}}(u)}\Delta u\left(\int_{t_1}^{\delta(t)}\frac{1}{r^{\frac{1}{\alpha}}(u)}\Delta u\right)^{-1}, \quad t\in[t_2,\infty)_{\mathbb{T}}, \end{aligned}$$

that is,

$$\frac{(z(\delta(t)))^{\beta}}{z^{\beta}(t)} \geq \psi^{\beta}(t,t_1), \quad t\in[t_2,\infty)_{\mathbb{T}}.$$

Thus, from (3.4) we have

$$\omega^{\Delta}(t) \leq -Q\varphi(t)\psi^{\beta}(t,t_1) + \frac{(\varphi^{\Delta}(t))_{+}}{\varphi^{\sigma}(t)}\omega^{\sigma}(t) - \varphi(t)\frac{\omega^{\sigma}(t)}{\varphi^{\sigma}(t)}\frac{(z^{\beta})^{\Delta}(t)}{z^{\beta}(t)}, \quad t\in[t_2,\infty)_{\mathbb{T}}. \tag{3.7}$$

Using $z^{\Delta}(t) > 0$ on $[t_1,\infty)_{\mathbb{T}}$ and Lemma 2.2, we obtain

$$\begin{aligned}\left(z^{\beta}(t)\right)^{\Delta} &= \beta z^{\Delta}(t)\int_0^1 \left[hz^{\sigma}(t) + (1-h)z(t)\right]^{\beta-1}\,dh \\ &\geq \begin{cases}\beta z^{\beta-1}(t)z^{\Delta}(t) & \text{if } \beta \geq 1, \\ \beta z^{\beta-1}(\sigma(t))z^{\Delta}(t) & \text{if } 0 < \beta < 1.\end{cases}\end{aligned} \tag{3.8}$$

From (3.7) and (3.8) we obtain, if $\beta \geq 1$,

$$\begin{aligned}\omega^{\Delta}(t) \leq{}& -Q(t)\varphi(t)\psi(t,t_1) + \frac{(\varphi^{\Delta}(t))_+}{\varphi^{\sigma}(t)}\omega^{\sigma}(t) \\ &- \beta\varphi(t)\frac{\omega^{\sigma}(t)}{\varphi^{\sigma}(t)}\frac{z^{\Delta}(t)}{z^{\sigma}(t)}\frac{z^{\sigma}(t)}{z(t)}, \quad t \in [t_2,\infty)_{\mathbb{T}}.\end{aligned} \tag{3.9}$$

If $0 < \beta < 1$ then

$$\begin{aligned}\omega^{\Delta}(t) \leq{}& -Q(t)\varphi(t)\psi(t,t_1) + \frac{(\varphi^{\Delta}(t))_+}{\varphi^{\sigma}(t)}\omega^{\sigma}(t) \\ &- \beta\varphi(t)\frac{\omega^{\sigma}(t)}{\varphi^{\sigma}(t)}\frac{z^{\Delta}(t)}{z^{\sigma}(t)}\left(\frac{z^{\sigma}(t)}{z(t)}\right)^{\beta}, \quad t \in [t_2,\infty)_{\mathbb{T}}.\end{aligned} \tag{3.10}$$

Since $z^{\Delta}(t) > 0$ and $r(t)(z^{\Delta}(t))^{\alpha}$ is strictly decreasing on $[t_1,\infty)_{\mathbb{T}}$, we have

$$z^{\sigma}(t) \geq z(t), \qquad z^{\Delta}(t) \geq \left(\frac{r^{\sigma}(t)}{r(t)}\right)^{\frac{1}{\alpha}}\left(z^{\Delta}(t)\right)^{\sigma}, \quad t \in [t_2,\infty)_{\mathbb{T}}. \tag{3.11}$$

Then, from (3.9), (3.10), (3.11) we can obtain

$$\begin{aligned}\omega^{\Delta}(t) \leq{}& -Q(t)\varphi(t)\psi(t,t_1) + \frac{(\varphi^{\Delta}(t))_+}{\varphi^{\sigma}(t)}\omega^{\sigma}(t) \\ &- \beta\varphi(t)\frac{\omega^{\sigma}(t)}{\varphi^{\sigma}(t)}\left(\frac{r^{\sigma}(t)}{r(t)}\right)^{\frac{1}{\alpha}}\frac{(z^{\Delta}(t))^{\sigma}}{z^{\sigma}(t)} \\ ={}& -Q(t)\varphi(t)\psi(t,t_1) + \frac{(\varphi^{\Delta}(t))_+}{\varphi^{\sigma}(t)}\omega^{\sigma}(t) \\ &- \frac{\beta\varphi(t)(z^{\frac{\beta-\alpha}{\alpha}}(t))^{\sigma}}{r^{\frac{1}{\alpha}}(t)(\varphi^{\sigma}(t))^{\frac{\alpha+1}{\alpha}}}\left(\omega^{\sigma}(t)\right)^{1+\frac{1}{\alpha}}, \quad t \in [t_2,\infty)_{\mathbb{T}}.\end{aligned}$$

We know that if $z(t) > 0$ is strictly increasing on $[t_1,\infty)$ and $\beta \geq \alpha$, then there exists a positive constant M such that $(z^{\frac{\beta-\alpha}{\alpha}})^{\sigma}(t) \geq M$. Hence, we have

$$\begin{aligned}\omega^{\Delta}(t) \leq{}& -Q(t)\varphi(t)\psi(t,t_1) + \frac{(\varphi^{\Delta}(t))_+}{\varphi^{\sigma}(t)}\omega^{\sigma}(t) \\ &- \frac{\beta M\varphi(t)}{r^{\frac{1}{\alpha}}(t)(\varphi^{\sigma}(t))^{\frac{\alpha+1}{\alpha}}}\left(\omega^{\sigma}(t)\right)^{\frac{\alpha+1}{\alpha}}, \quad t \in [t_2,\infty)_{\mathbb{T}}.\end{aligned}$$

Letting $B = \frac{(\varphi^{\Delta}(t))_+}{\varphi^{\sigma}(t)}$, $A = \frac{\beta M\varphi(t)}{r^{\frac{1}{\alpha}}(t)(\varphi^{\sigma}(t))^{\frac{\alpha+1}{\alpha}}}$, $u = \omega^{\sigma}$, and using the inequality

$$Bu - Au^{\frac{\alpha+1}{\alpha}} \leq \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{B^{\alpha+1}}{A^{\alpha}}, \quad A > 0,$$

we have

$$\omega^{\Delta}(t) \le -Q(t)\varphi(t)\psi^{\beta}(t,t_1) + \frac{\alpha^{\alpha} r(t)((\varphi^{\Delta}(t))_+)^{\alpha+1}}{(\alpha+1)^{\alpha+1}(\beta M)^{\alpha}\varphi^{\alpha}(t)}. \tag{3.12}$$

Integrating both sides of (3.12) from t_2 to t, since $\omega(t) > 0$ for $t \in [t_2,\infty)_{\mathbb{T}}$, we obtain

$$\int_{t_2}^{t}\left[Q(s)\varphi(s)\psi^{\beta}(s,t_1) - \frac{\alpha^{\alpha} r(s)((\varphi^{\Delta}(s))_+)^{\alpha+1}}{(\alpha+1)^{\alpha+1}(\beta M)^{\alpha}\varphi^{\alpha}(s)}\right]\Delta s \le \omega(t_2) - \omega(t) < \omega(t_2),$$

which contradicts (3.1). The proof is complete. □

Remark 3.1 From theorem given in this section, we can get Philos-type oscillation criteria for equation (1.1) easily. The details are left to the reader.

Remark 3.2 From theorem obtained in this section, we can get various oscillation criteria of equation (1.1) by different choices of $\varphi(t)$.

For example, let $\varphi(s) = s$. We can get the following results from Theorem 3.1.

Corollary 3.1 *Assume that* (A1)-(A4) *hold*, $0 \le p(t) \le 1$ *and* $\tau(t) \le t$, $\delta(t) \le t$ *for* $t \in [t_0,\infty)_{\mathbb{T}}$. *If for any positive number* M *and sufficiently large* $t_1 \ge t_0$ *we have*

$$\limsup_{t\to\infty}\int_{t_2}^{t}\left[sQ(s)\psi^{\beta}(s,t_1) - \frac{\alpha^{\alpha} r(s)}{(\alpha+1)^{\alpha+1}(\beta M s)^{\alpha}}\right]\Delta s = \infty,$$

where $t_2 > t_1$ *such that* $\delta(t) > t_1$ *for* $t \in [t_2,\infty)_{\mathbb{T}}$, $Q(s)$ *and* $\psi(s,t_1)$ *are defined as in Theorem* 3.1. *Then every solution of* (1.1) *is oscillatory.*

Let $\varphi(s) = 1$. Then from Theorem 3.1, we have the following results.

Corollary 3.2 *Assume that* (A1)-(A4) *hold*, $0 \le p(t) \le 1$ *and* $\tau(t) \le t$, $\delta(t) \le t$ *for* $t \in [t_0,\infty)_{\mathbb{T}}$. *If for any positive number* M *and sufficiently large* $t_1 \ge t_0$ *we have*

$$\limsup_{t\to\infty}\int_{t_2}^{t} Q(s)\psi^{\beta}(s,t_1)\Delta s = \infty,$$

where $t_2 > t_1$ *such that* $\delta(t) > t_1$ *for* $t \in [t_2,\infty)_{\mathbb{T}}$; $Q(s)$ *and* $\psi(s,t_1)$ *are defined as in Theorem* 3.1. *Then every solution of* (1.1) *is oscillatory.*

4 Oscillation of equation (1.1) when $p(t) > 1$

In this section, we will use the following notation:

τ^{-1} is the inverse function of τ;

$\eta^{\Delta}(t)_+ := \max\{0,\eta^{\Delta}(t)\}$, $\gamma(t) := \begin{cases} \frac{m(t)}{m^{\sigma}(t)} & \text{if } \beta < 1, \\ (\frac{m(t)}{m^{\sigma}(t)})^{\beta} & \text{if } \beta \ge 1; \end{cases}$

$p^*(t) := \frac{1}{p(\tau^{-1}(t))}(1 - \frac{1}{p(\tau^{-1}(\tau^{-1}(t)))}) > 0$;

$p_*(t) := \frac{1}{p(\tau^{-1}(t))}(1 - \frac{1}{p(\tau^{-1}(\tau^{-1}(t)))}\frac{m(\tau^{-1}(\tau^{-1}(t)))}{m(\tau^{-1}(t))}) > 0$, for all sufficiently large t, where m will be specified later.

Theorem 4.1 *Assume that* (A1)-(A4) *hold, and let τ be strictly increasing, $\tau(t) > t$ and $\tau(\sigma(t)) \geq \delta(t)$. If there exists a positive function $\eta, m \in C^1_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ such that*

$$\frac{m(t)}{r^{\frac{1}{\alpha}}(t)\int_{t_1}^{t} r^{-\frac{1}{\alpha}}(s)\Delta s} - m^{\Delta}(t) \leq 0, \tag{4.1}$$

for all sufficiently large $t \geq t_1 \geq t_0$, and for some $t_2 \in [t_1,\infty)_{\mathbb{T}}$ and any positive constant M, one has

$$\begin{aligned}\limsup_{t\to\infty}\int_{t_2}^{t}\biggl[&\eta^{\sigma}(s)q(s)\bigl(p^*(\delta(s))\bigr)^{\beta}\left(\frac{m(\tau^{-1}(\delta(s)))}{m^{\sigma}(s)}\right)^{\beta}\\ &-\frac{\alpha^{\alpha}r(s)((\eta^{\Delta}(s))_+)^{\alpha+1}}{(\alpha+1)^{\alpha+1}(\beta M\gamma(s)\eta^{\sigma}(s))^{\alpha}}\biggr]\Delta s=\infty,\end{aligned} \tag{4.2}$$

then every solution of (1.1) *is oscillatory.*

Proof Let x be a nonoscillatory solution of (1.1). Without loss of generality, we may assume that there exists $t_1 \in [t_0,\infty)_{\mathbb{T}}$ such that $x(t) > 0, x(\tau(t)) > 0, x(\delta(t)) > 0$ for $t \in [t_1,\infty)_{\mathbb{T}}$. Then $z^{\Delta}(t) > 0$ for $t \in [t_1,\infty)_{\mathbb{T}}$ due to Lemma 2.1. From $x(\tau(t)) = \frac{1}{p(t)}(z(t) - x(t))$, it follows that

$$\begin{aligned}x(t) &= \frac{1}{p(\tau^{-1}(t))}\bigl(z\bigl(\tau^{-1}(t)\bigr) - x\bigl(\tau^{-1}(t)\bigr)\bigr)\\ &= \frac{z(\tau^{-1}(t))}{p(\tau^{-1}(t))} - \frac{1}{p(\tau^{-1}(t))}\left(\frac{z(\tau^{-1}(\tau^{-1}(t)))}{p(\tau^{-1}(\tau^{-1}(t)))} - \frac{x(\tau^{-1}(\tau^{-1}(t)))}{p(\tau^{-1}(\tau^{-1}(t)))}\right)\\ &\geq \frac{z(\tau^{-1}(t))}{p(\tau^{-1}(t))} - \frac{z(\tau^{-1}(\tau^{-1}(t)))}{p(\tau^{-1}(t))p(\tau^{-1}(\tau^{-1}(t)))}\\ &\geq \frac{1}{p(\tau^{-1}(t))}\left(1 - \frac{1}{p(\tau^{-1}(\tau^{-1}(t)))}\right)z\bigl(\tau^{-1}(t)\bigr)\\ &= p^*(t)z\bigl(\tau^{-1}(t)\bigr).\end{aligned}$$

From this and (1.1), we have

$$\bigl(r(t)\bigl(z^{\Delta}(t)\bigr)^{\alpha}\bigr)^{\Delta} + q(t)\bigl(p^*\bigl(\delta(t)\bigr)\bigr)^{\beta}\bigl(z\bigl(\tau^{-1}\bigl(\delta(t)\bigr)\bigr)\bigr)^{\beta} \leq 0. \tag{4.3}$$

On the other hand, we have

$$z(t) = z(t_1) + \int_{t_1}^{t}\frac{(r(s)(z^{\Delta}(s))^{\alpha})^{\frac{1}{\alpha}}}{r^{\frac{1}{\alpha}}(s)}\Delta s \geq \left(r^{\frac{1}{\alpha}}(t)\int_{t_1}^{t}\frac{1}{r^{\frac{1}{\alpha}}(s)}\Delta s\right)z^{\Delta}(t).$$

From (2.2) and (4.1), we have

$$\begin{aligned}\left(\frac{z}{m}\right)^{\Delta}(t) &= \frac{z^{\Delta}(t)m(t) - z(t)m^{\Delta}(t)}{m(t)m^{\sigma}(t)}\\ &\leq \frac{z(t)}{m(t)m^{\sigma}(t)}\left(\frac{m(t)}{r^{\frac{1}{\alpha}}(t)\int_{t_1}^{t} r^{-\frac{1}{\alpha}}(s)\Delta s} - m^{\Delta}(t)\right) \leq 0.\end{aligned}$$

Hence $\frac{z}{m}$ is a nonincreasing function. Since $\tau^{-1}(\delta(t)) \leq \sigma(t)$ and $t \leq \sigma(t)$, we obtain

$$\frac{z(\tau^{-1}(\delta(t)))}{z^{\sigma}(t)} \geq \frac{m(\tau^{-1}(\delta(t)))}{m^{\sigma}(t)}, \qquad \frac{z(t)}{z^{\sigma}(t)} \geq \frac{m(t)}{m^{\sigma}(t)}. \tag{4.4}$$

Define a function

$$\omega(t) = \eta(t)\frac{r(t)(z^{\Delta}(t))^{\alpha}}{(z(t))^{\beta}}, \quad t \in [t_1, \infty)_{\mathbb{T}}. \tag{4.5}$$

Obviously, $\omega(t) > 0$. From (2.1), (2.2), (4.3), and (4.5), we have

$$\begin{aligned} \omega^{\Delta}(t) &= \eta^{\Delta}\frac{r(z^{\Delta})^{\alpha}}{z^{\beta}} + \eta^{\sigma}\frac{(r(z^{\Delta})^{\alpha})^{\Delta}z^{\beta} - r(z^{\Delta})^{\alpha}(z^{\beta})^{\Delta}}{z^{\beta}(z^{\beta})^{\sigma}} \\ &\leq \frac{(\eta^{\Delta})_{+}}{\eta}\omega - \eta^{\sigma}q\bigl((p\circ\delta)^{*}\bigr)^{\beta}\left(\frac{z(\tau\circ\delta)^{-1}}{z^{\sigma}}\right)^{\beta} - \eta^{\sigma}\frac{r(z^{\Delta})^{\alpha}(z^{\beta})^{\Delta}}{z^{\beta}(z^{\beta})^{\sigma}}. \end{aligned} \tag{4.6}$$

By (4.4), (4.5), (4.6), and (3.8), we can get, if $\beta \geq 1$, then

$$\begin{aligned} \omega^{\Delta}(t) \leq{}& \frac{(\eta^{\Delta}(t))_{+}}{\eta(t)}\omega(t) - \eta^{\sigma}(t)q(t)\bigl(p^{*}\bigl(\delta(t)\bigr)\bigr)^{\beta}\left(\frac{m(\tau^{-1}(\delta(t)))}{m^{\sigma}(t)}\right)^{\beta} \\ &- \beta\frac{\eta^{\sigma}(t)}{r^{\frac{1}{\alpha}}(t)\eta^{\frac{\alpha+1}{\alpha}}(t)}z^{\frac{\beta-\alpha}{\alpha}}(t)\omega^{\frac{1+\alpha}{\alpha}}(t)\left(\frac{m(t)}{m^{\sigma}(t)}\right)^{\beta}, \end{aligned} \tag{4.7}$$

and if $0 < \beta < 1$, then

$$\begin{aligned} \omega^{\Delta}(t) \leq{}& \frac{(\eta^{\Delta}(t))_{+}}{\eta(t)}\omega(t) - \eta^{\sigma}(t)q(t)\bigl(p^{*}\bigl(\delta(t)\bigr)\bigr)^{\beta}\left(\frac{m(\tau^{-1}(\delta(t)))}{m^{\sigma}(t)}\right)^{\beta} \\ &- \beta\frac{\eta^{\sigma}(t)}{r^{\frac{1}{\alpha}}(t)\eta^{\frac{\alpha+1}{\alpha}}(t)}z^{\frac{\beta-\alpha}{\alpha}}(t)\omega^{\frac{1+\alpha}{\alpha}}(t)\frac{m(t)}{m^{\sigma}(t)}. \end{aligned} \tag{4.8}$$

Since $z(t)$ is strictly increasing on $[t_1, \infty)_{\mathbb{T}}$ and $\beta \geq \alpha$, there exists a positive constant M such that $z^{\frac{\beta-\alpha}{\alpha}} \geq M$. Combining with (4.7), (4.8), and the definition of $\gamma(t)$, we know that

$$\begin{aligned} \omega^{\Delta}(t) \leq{}& \frac{(\eta^{\Delta}(t))_{+}}{\eta(t)}\omega(t) - \eta^{\sigma}(t)q(t)\bigl(p^{*}\bigl(\delta(t)\bigr)\bigr)^{\beta}\left(\frac{m(\tau^{-1}(\delta(t)))}{m^{\sigma}(t)}\right)^{\beta} \\ &-\beta M\gamma(t)\frac{\eta^{\sigma}(t)}{r^{\frac{1}{\alpha}}(t)\eta^{\frac{\alpha+1}{\alpha}}(t)}\omega^{\frac{1+\alpha}{\alpha}}(t) \end{aligned}$$

holds for $\beta > 0$. Letting $B = \frac{(\eta^{\Delta}(t))_{+}}{\eta(t)}$, $A = \beta M\gamma(t)\frac{\eta^{\sigma}(t)}{r^{\frac{1}{\alpha}}(t)\eta^{\frac{\alpha+1}{\alpha}}(t)}$, $u = \omega(t)$, and using the inequality

$$Bu - Au^{\frac{1+\alpha}{\alpha}} \leq \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{B^{\alpha+1}}{A^{\alpha}}, \quad A > 0,$$

we have

$$\omega^{\Delta}(t) \leq -\eta^{\sigma}(t)q(t)\bigl(p^{*}\bigl(\delta(t)\bigr)\bigr)^{\beta}\left(\frac{m(\tau^{-1}(\delta(t)))}{m^{\sigma}(t)}\right)^{\beta} + \frac{\alpha^{\alpha}r(t)((\eta^{\Delta}(t))_{+})^{\alpha+1}}{(\alpha+1)^{\alpha+1}(\beta M\gamma(t)\eta^{\sigma}(t))^{\alpha}}. \tag{4.9}$$

Integrating (4.9) from $t_2 \in [t_1, \infty)_{\mathbb{T}}$ to t, we have

$$\int_{t_2}^{t} \left[\eta^{\sigma}(s) q(s) \bigl(p^*\bigl(\delta(s)\bigr)\bigr)^{\beta} \left(\frac{m(\tau^{-1}(\delta(s)))}{m^{\sigma}(s)} \right)^{\beta} - \frac{\alpha^{\alpha} r(s)((\eta^{\Delta}(s))_+)^{\alpha+1}}{(\alpha+1)^{\alpha+1}(\beta M \gamma(s) \eta^{\sigma}(s))^{\alpha}} \right] \Delta s \le \omega(t_2),$$

which contradicts (4.2). The proof is complete. □

Theorem 4.2 *Assume that* (A1)-(A4) *hold, and let* τ *be strictly increasing,* $\tau(t) > t$ *and* $\tau(\sigma(t)) \le \delta(t)$*. If there exists a positive function* $\eta, m \in C^1_{\mathrm{rd}}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$ *such that* (4.1) *holds for all sufficiently large* t_1*, and for some* $t_2 \in [t_1, \infty)_{\mathbb{T}}$ *and any positive constant* M*, one has*

$$\limsup_{t \to \infty} \int_{t_2}^{t} \left[\eta\bigl(\sigma(s)\bigr) q(s) \bigl(p^*\bigl(\delta(s)\bigr)\bigr)^{\beta} - \frac{\alpha^{\alpha} r(s)((\eta^{\Delta}(s))_+)^{\alpha+1}}{(\alpha+1)^{\alpha+1}(\beta M \gamma(s) \eta(\sigma(s)))^{\alpha}} \right] \Delta s = \infty, \tag{4.10}$$

then every solution of (1.1) *is oscillatory.*

Proof Proceeding as in the proof of Theorem 4.1, we have (4.6). Since $\tau(\sigma(t)) \le \delta(t)$ and z is strictly increasing on $[t_1, \infty)_{\mathbb{T}}$, we obtain $\tau^{-1}(\delta(t)) \ge \sigma(t)$ and $\frac{z(\tau^{-1}(\delta(t)))}{z(\sigma(t))} \ge 1$. Hence

$$\omega^{\Delta}(t) \le \frac{(\eta^{\Delta}(t))_+}{\eta(t)} \omega(t) - \eta\bigl(\sigma(t)\bigr) q(t) \bigl(p^*\bigl(\delta(t)\bigr)\bigr)^{\beta} - \eta\bigl(\sigma(t)\bigr) \frac{r(t)(z^{\Delta}(t))^{\alpha}(z^{\beta}(t))^{\Delta}}{z^{\beta}(t) z^{\beta}(\sigma(t))}.$$

The remainder of the proof is similar to that of Theorem 4.1, we can get a contradiction to (4.10). This completes the proof. □

Theorem 4.3 *Assume that* (A1)-(A4) *hold, and let* τ *be strictly increasing,* $\tau(t) < t$ *and* $\tau(\sigma(t)) \ge \delta(t)$*. If there exists a positive function* $\eta, m \in C^1_{\mathrm{rd}}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$ *such that* (4.1) *holds for all sufficiently large* t_1*, and for some* $t_2 \in [t_1, \infty)_{\mathbb{T}}$ *and any positive constant* M*, one has*

$$\begin{aligned} \limsup_{t \to \infty} \int_{t_2}^{t} & \left[\eta\bigl(\sigma(s)\bigr) q(s) \bigl(p_*\bigl(\delta(s)\bigr)\bigr)^{\beta} \left(\frac{m(\tau^{-1}(\delta(s)))}{m(\sigma(s))} \right)^{\beta} \right. \\ & \left. - \frac{\alpha^{\alpha} r(s)((\eta^{\Delta}(s))_+)^{\alpha+1}}{(\alpha+1)^{\alpha+1}(\beta M \gamma(s) \eta(\sigma(s)))^{\alpha}} \right] \Delta s = \infty, \end{aligned} \tag{4.11}$$

then every solution of (1.1) *is oscillatory.*

Proof Proceeding as in the proof of Theorem 4.1, we know that $\frac{z}{m}$ is nonincreasing. Since $\tau^{-1}(\tau^{-1}(t)) \ge \tau^{-1}(t)$, we obtain $\frac{z(\tau^{-1}(t)) m(\tau^{-1}(\tau^{-1}(t)))}{m(\tau^{-1}(t))} \ge z(\tau^{-1}(\tau^{-1}(t)))$, then we have

$$\begin{aligned} x(t) &= \frac{1}{p(\tau^{-1}(t))} \bigl(z\bigl(\tau^{-1}(t)\bigr) - x\bigl(\tau^{-1}(t)\bigr)\bigr) \\ &\ge \frac{z(\tau^{-1}(t))}{p(\tau^{-1}(t))} - \frac{z(\tau^{-1}(\tau^{-1}(t)))}{p(\tau^{-1}(t)) p(\tau^{-1}(\tau^{-1}(t)))} \\ &\ge \frac{z(\tau^{-1}(t))}{p(\tau^{-1}(t))} - \frac{1}{p(\tau^{-1}(t)) p(\tau^{-1}(\tau^{-1}(t)))} \frac{z(\tau^{-1}(t)) m(\tau^{-1}(\tau^{-1}(t)))}{m(\tau^{-1}(t))} \\ &= p_*(t) z\bigl(\tau^{-1}(t)\bigr). \end{aligned}$$

The remainder of the proof is similar to that of Theorem 4.1 and we can get a contradiction to (4.11). This completes the proof. □

Remark 4.1 From the theorems obtained in this section, we can get various oscillation criteria of equation (1.1) by different choices of $m(t)$ and $\eta(t)$.

For example, let $\eta(t) = 1$ and $m(t) = \int_{t_1}^{t} r^{-\frac{1}{\alpha}}(s)\Delta s$. We obtain the following results from Theorem 4.1.

Corollary 4.1 *Assume that* (A1)-(A4) *hold, and let* τ *be strictly increasing,* $\tau(t) > t$ *and* $\tau(\sigma(t)) \geq \delta(t)$. *If for all sufficiently large* t_1, *and for some* $t_2 \in [t_1, \infty)_{\mathbb{T}}$, *one has*

$$\limsup_{t\to\infty} \int_{t_2}^{t} q(s)\bigl(p^*\bigl(\delta(s)\bigr)\bigr)^{\beta}\left(\frac{m(\tau^{-1}(\delta(s)))}{m^{\sigma}(s)}\right)^{\beta} \Delta s = \infty,$$

then every solution of (1.1) *is oscillatory.*

5 Examples

In this section, we will give some examples in order to illustrate our main results of this paper.

Example 5.1 Consider the equation

$$\bigl(\bigl|z^{\Delta}(t)\bigr|z^{\Delta}(t)\bigr)^{\Delta} + t\left|x\left(\frac{t}{2}\right)\right|^{2} = 0, \quad t \in [0, +\infty)_{\mathbb{T}}, \tag{5.1}$$

where $z(t) = x(t) + \frac{1}{2}x(\frac{t}{3})$.

In equation (5.1), $\tau(t) = \frac{t}{3}$, $\delta(t) = \frac{t}{2}$, $\alpha = 2$, $\beta = 3$, $r(t) = 1$, $p(t) = \frac{1}{2}$, $q(t) = t$, take $\tau(t) < t$, $\delta(t) < t$, and (A1)-(A4) to hold. Choose $\varphi(t) = 1$. For any given $t_1 > 0$, we have

$$Q(s) = q(s)\bigl[1 - p\bigl(\delta(s)\bigr)\bigr]^{\beta} = \frac{s}{8},$$

$$\psi(s, t_1) = \left(\int_{t_1}^{s} \frac{1}{r^{\frac{1}{\alpha}}(u)}\Delta u\right)^{-1} \int_{t_1}^{\delta(s)} \frac{1}{r^{\frac{1}{\alpha}}(u)}\Delta u = \frac{\frac{s}{2} - t_1}{s - t_1} \geq \frac{1}{3}, \quad \text{for } s \geq 4t_1.$$

Thus as $t_2 > 4t_1$ we get

$$\limsup_{t\to\infty} \int_{t_2}^{t} \left[\varphi(s)Q(s)\psi^{\beta}(s, t_1) - \frac{\alpha^{\alpha} r(s)((\varphi^{\Delta}(s))_{+})^{\alpha+1}}{(\alpha+1)^{\alpha+1}(\beta M)^{\alpha}\varphi^{\alpha}(s)}\right]\Delta s = \infty.$$

Therefore, condition (3.1) holds, and hence equation (5.1) is oscillatory due to Theorem 3.1.

Example 5.2 Consider the equation

$$z^{\Delta\Delta}(t) + \frac{\sigma(t)}{t^2}x(t) = 0, \quad t \in [1, \infty) \cap \mathbb{T}, \tag{5.2}$$

where $z(t) = x(t) + 2x(2t)$.

In equation (5.2), $\alpha = \beta = 1$, $r(t) = 1$, $p(t) = 2$, $q(t) = \frac{\sigma(t)}{t^2}$, $\tau(t) = 2t$, $\delta(t) = t$, τ is a strictly increasing function and $\tau(\sigma(t)) \geq \delta(t)$ and (A1)-(A4) hold. Choosing $\eta(t) = 1$ and $m(t) = t^2$, then we have (4.1) for $t \geq 2t_1 \geq 2$. In order to using Theorem 4.1, we need to show that (4.2) holds. In fact,

$$q(s)\big(p^*\big(\delta(s)\big)\big)^\beta \left(\frac{m(\tau^{-1}(\delta(s)))}{m^\sigma(s)}\right)^\beta = \frac{\sigma(s)}{4s^2} \times \frac{s^2}{4\sigma^2(s)} = \frac{1}{16\sigma(s)}.$$

Using Theorem 5.59 of [2], we can obtain

$$\limsup_{t\to\infty} \int_{t_2}^{t} q(s)\big(p^*\big(\delta(s)\big)\big)^\beta \left(\frac{m(\tau^{-1}(\delta(s)))}{m^\sigma(s)}\right)^\beta \Delta s = \infty, \quad t_2 \geq 2t_1.$$

Therefore, condition (4.2) holds, and hence equation (5.2) is oscillatory due to Theorem 4.1.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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